New integrable quantum chains combining different kinds of spins

1992 J. Phys. A: Math. Gen. 25 4499

(http://iopscience.iop.org/0305-4470/25/17/012)

J. Phys. A: Math. Gen. 25 (1992) 4499-4516. Printed in the UK

New integrable quantum chains combining different kinds of spins

H J de Vega† and F Woynarovich‡

† Laboratoire de Physique Théorique et Hautes Energies§, Tour 16, 1er étage, *Université Paris VI, 4 place Jussieu, 75252 Paris Cedex 05, France*
‡ Central Research Institue for Physics, H-1525 Budapest 114, POB 49, Hungary

Received 5 January 1992

Abstract. A general procedure to generate new integrable Hamiltonians combining any kind of spins distributed arbitrarily on the line is given. As a concrete application, anisotropic chains formed by spin-$\frac{1}{2}$ and spin-1 operators at alternating sites are presented and solved exactly by Bethe ansatz (BA). We compute the ground-state and excitation energies and momentum. The higher-order BA equations are derived. Depending on our choice, these new Hamiltonians exhibit or not conformal invariance in their low energy spectrum.

1. Introduction

Integrable magnetic chains are interesting physical systems with a rich mathematical structure. The best-known by far is the XXZ Heisenberg chain with $S=\frac{1}{2}$ spins [1]. Integrable spin-1 [2] and higher spin [3] chains have been found and solved. In addition magnetic Hamiltonians can be derived from Yang–Baxter (YB) solutions associated with Lie algebras other than SU(2) [4, 5].

The purpose of this paper is to present and solve integrable magnetic chains formed by *two* kind of spins or more. The simplest case is an alternating chain with spin-$\frac{1}{2}$ and spin-1 operators. The Hamiltonians are derived from appropriately chosen solutions of the YB equation.

We find and solve two integrable Hamiltonians describing spin-$\frac{1}{2}$ and spin-1 operators at alternative sites. We call them $\bar{H}(\alpha)$ and $\tilde{H}(\alpha)$. $\tilde{H}(\alpha)$ is given explicitly by equations (3.10)–(3.13). They contain a piece coupling pairs of neighbouring spin-$\frac{1}{2}$ and 1 operators and another piece coupling three neighbouring spins. For $\tilde{H}$ two spins-$\frac{1}{2}$ and one spin-1 and for $\bar{H}$ two spins-1 and one spin-$\frac{1}{2}$. These Hamiltonians are invariant under rotations around the z-axis and depend on two arbitrary continuous parameters α and $\gamma\cdot\gamma$ is connected with the $\mathrm{SU}(2)_q$ parameter by $q=e^{i\gamma}$ or $e^{-\gamma}$. Depending on the choice of γ and α we find ferromagnetic or antiferromagnetic behaviours. This leads to a gapless regime (in the weak antiferromagnetic case) or a non-zero gap regime (in the ferromagnetic or strong antiferromagnetic cases). Notice that the gap vanishes or not irrespective of the kind of spins contained in the Hamiltonian.

§ Unité Associée au CNRS, UA 280.

More general integrable magnetic Hamiltonians can be easily defined. Possible generalizations are:

(a) To consider several kinds of spins $s_1, s_2, \ldots$.

(b) Spins of different kind may be distributed arbitrarily on the line. That is, the spin values do not need to alternate from site to site. *Any* distribution leads to an integrable chain.

(c) Include operators linked to Lie algebras other than A_1. That is, Hamiltonian-based on Yang–Baxter solutions associated to a Lie algebra $\mathscr{G} \neq A_1$ and acting on two different representation spaces.

In section 4 we solve exactly the spin-$\frac{1}{2}$–spin-1 mixed Hamiltonian $\bar{H}(\alpha)$ and $\tilde{H}(\alpha)$ by Bethe ansatz. We obtain their ground-state energies and the elementary excitation spectrum in section 5. The ground state is formed by a distribution of real roots and a distribution of (complex) 2-strings. Holes on the real root distribution describe excitations with non-zero energy for $\tilde{H}(\alpha)$ and zero energy for $\bar{H}(\alpha)$. The converse is true for holes in the 2-string distribution.

We give the combination of the the two Hamiltonians in which both kinds of excitations have the same dispersion law and hence the system is conformally invariant. We conclude by checking the string hypothesis for the complex roots in these new models and we derive the higher-level Bethe ansatz equations.

2. Integrable quantum chains with two types of spins

As is known, regular solutions of the Yang–Baxter equations systematically yield integrable spin chains [2-6]. Let us briefly review how a spin Hamiltonian follows from an $\bar{R}$-matrix $\bar{R}^{\alpha\beta}_{\gamma\delta}(\theta)$, which is a solution of the Yang–Baxter equation (YBE)

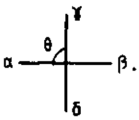
$$[1\otimes R(\theta-\theta')][R(\theta)\otimes 1][1\otimes R(\theta')]=[R(\theta')\otimes 1][1\otimes R(\theta)][R(\theta-\theta')\otimes 1]. \tag{2.1}$$

We consider $2N$ sites in a row and we associate to it the operator

$$T_{\alpha\beta}(\theta)=\sum_{\alpha_1\ldots\alpha_{2N-1}} t^{(1)}_{\alpha\alpha_1}(\theta)\otimes t^{(2)}_{\alpha_1\alpha_2}(\theta)\otimes\ldots\otimes t^{(2N)}_{\alpha_{2N-1}\beta}(\theta) \tag{2.2}$$

where $t^{(i)}_{\alpha\beta}(\theta)$ acts on the spin space of the site i and

$$[t_{\alpha\beta}(\theta)]_{\gamma\delta}=R^{\beta\delta}_{\gamma\alpha}(\theta)= \tag{2.3}$$



Thanks to the YBE (2.1), $t_{\alpha\beta}(\theta)$ and $T_{\alpha\beta}(\theta)$ obey also a YBE

$$R(\theta-\theta')[t(\theta)\otimes t(\theta')]=[t(\theta')\otimes t(\theta)]R(\theta-\theta') \tag{2.4a}$$

$$R(\theta-\theta')[T(\theta)\otimes T(\theta')]=[T(\theta')\otimes T(\theta)]R(\theta-\theta'). \tag{2.4b}$$

Due to (2.4), the transfer matrices

$$\tau(\theta)=\sum_{\alpha} T_{\alpha\alpha}(\theta) \tag{2.5}$$

form a commuting family

$$[\tau(\theta), \tau(\theta')]=0. \tag{2.6}$$

In most cases

$$R(0)=c\mathbb{1} \tag{2.7}$$

and then (2.1) implies that [4]

$$R(\theta)R(-\theta)=\rho(\theta)1 \tag{2.8}$$

where $\rho(\theta)=\rho(-\theta)$ is a c-number function. Equation (2.7) implies that the transfer matrix (2.5) at $\theta=0$ equals the unit shift operator

$$[\tau(0)]_{\underset{\sim}{\alpha}|\underset{\sim}{\beta}}=c^{2N}\prod_{i=1}^{2N}\delta_{\alpha_i\beta_{i+1}} \tag{2.9}$$

and we used

$$\begin{aligned}&\underset{\sim}{\alpha}=(\alpha_1,\ldots,\alpha_{2N}) \qquad \underset{\sim}{\beta}=(\beta_1,\ldots,\beta_{2N}) \quad \text{and} \quad \beta_{2N+1}=\beta_1\\ &[t_{\alpha\beta}(0)]_{\gamma\delta}=c\delta_{\alpha\delta}\delta_{\beta\gamma}.\end{aligned} \tag{2.10}$$

As a consequence of (2.9) the logarithmic derivative of $\tau(\theta)$ at $\theta=0$ gives an operator coupling pairs of nearest neighbours. One finds [4]

$$\frac{\partial}{\partial\theta}\ln\tau(\theta)|_{\theta=0}=\sum_{M=1}^{2N}h_{M,M+1} \tag{2.11}$$

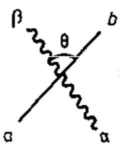
where

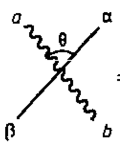
$$[h_{M,M+1}]_{\alpha_M\alpha_{M+1}|\beta_M\beta_{M+1}}=\frac{1}{c}\dot{R}(0)^{\alpha_M\alpha_{M+1}}_{\beta_M\beta_{M+1}}. \tag{2.12}$$

Clearly, the operator (2.11) can be interpreted as a one-dimensional quantum Hamiltonian. It couples neighbouring spins. For the six-vertex (eight-vertex) R-matrix one obtains through equations (2.11)-(2.12) the XXZ (XYZ) Hamiltonian.

Equation (2.1) is not the most general YBE. In general we may have YB operators acting on pairs of unequal vector spaces. This corresponds graphically to lines of different kind.

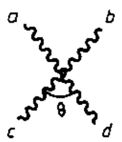
That is, we have the operators



$=\{\tilde{t}_{\beta\alpha}(\theta)\}_{ba}$


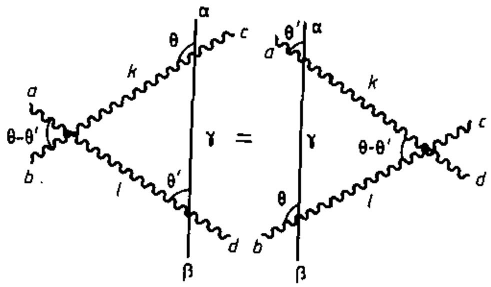
$=[\bar{t}_{ab}(\theta)]_{\alpha\beta}.$ (2.13)

Here $1\leq\alpha,\ \beta\leq q_1$ (lines ______) and $1\leq a,\ b\leq q_2$ (lines ∿∿∿∿). $\tilde{t}_{\alpha\beta}(\theta)$ also fulfils (2.4a). In addition, (2.1) holds for the R-matrix

$\bar{R}^{ab}_{cd}(\theta)=$


. (2.14)

Finally



(2.15)

$$\bar{R}(\theta-\theta')[\tilde{t}(\theta)\otimes\tilde{t}(\theta')]=[\tilde{t}(\theta')\otimes\tilde{t}(\theta)]\bar{R}(\theta-\theta').$$

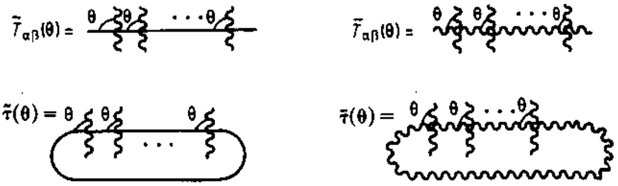
In addition, we assume T-invariance (symmetry) for the R-matrices and t-operators

$$R^{\alpha\beta}_{\gamma\delta}(\theta)=R^{\gamma\delta}_{\alpha\beta}(\theta) \qquad \bar{R}^{ab}_{cd}(\theta)=\bar{R}^{cd}_{ab}(\theta)$$
$$[\tilde{t}_{\beta\alpha}(\theta)]_{ba}=[\tilde{t}_{ab}(\theta)]_{\alpha\beta}. \tag{2.16}$$

Provided $\bar{R}(0)=\bar{c}\mathbb{1}$, we have [4] the 'unitary' relation

$$\sum_{\beta b}[\tilde{t}_{ab}(\theta)]_{\alpha\beta}[\tilde{t}_{bc}(-\theta)]_{\beta\gamma}=\delta_{ac}\delta_{\alpha\gamma}\tilde{\rho}(\theta). \tag{2.17}$$

Further YB operators $\tilde{T}_{\alpha\beta}(\theta)$, $\bar{T}_{\alpha\beta}(\theta)$ and commuting transfer matrices $\tilde{\tau}(\theta)$ and $\bar{\tau}(\theta)$ are constructed as follows



(2.18)

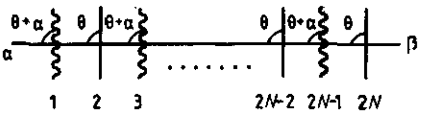
$\bar{\tau}(\theta)$ generates local quantum Hamiltonians coupling nearest neighbours as $\tau(\theta)$ does (2.10), (2.11)). This is not the case for $\tau(\theta)$, since

$$[\tilde{t}_{ab}(0)]_{\alpha\beta}$$

just cannot produce deltas as $[t_{\alpha\beta}(\theta)]_{\gamma\delta}$ does in (2.10). An operator connecting different spaces (______ and ∿∿∿∿) cannot be a unit operator.

In conclusion $\tilde{\tau}(\theta)$ generates operators coupling *all* spins in the chain. However, $\tilde{\tau}_{\alpha\beta}(\theta)$ is not the most general operator obeying the YBE (2.4*b*) that we can build in the present context. Let us consider

$\tilde{T}^{(\mathrm{alt})}_{\alpha\beta(\theta,\alpha}=$



$$=\sum_{\alpha_1\ldots\alpha_{2N-1}}\tilde{t}^{(1)}_{\alpha\alpha_1}(\theta+\alpha)t^{(2)}_{\alpha_1\alpha_2}(\theta)\tilde{t}^{(3)}_{\alpha_2\alpha_3}(\theta+\alpha)$$
$$\ldots\tilde{t}^{(2N-1)}_{\alpha_{2N-2}\alpha_{2N-1}}(\theta+\alpha)t^{(2N)}_{\alpha_{2N-1}\beta}(\theta). \tag{2.19}$$

Notice that ∿∿∿∿ lines and ______ lines sit at odd and even sites, respectively.

This operator fulfils also (2.4*b*) for fixed α

$$R(\theta-\theta')[\tilde{T}^{(\mathrm{alt})}(\theta,\alpha)\otimes\tilde{T}^{(\mathrm{alt})}(\theta',\alpha)]=[\tilde{T}^{(\mathrm{net})}(\theta',\alpha)\otimes\tilde{T}^{(\mathrm{net})}(\theta,\alpha)]R(\theta-\theta'). \tag{2.20}$$

Of course, much more general operators fulfilling the YBE can be constructed (see discussion at the end of this section and at the end of section 3).

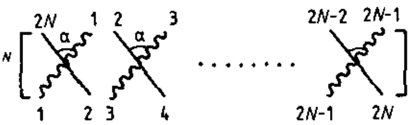
A commuting family of transfer matrices

$$\tilde{\tau}^{(\mathrm{alt})}(\theta,\alpha)=\sum_{\alpha}\tilde{T}^{(\mathrm{net})}\alpha\alpha(\theta,\alpha) \tag{2.21}$$

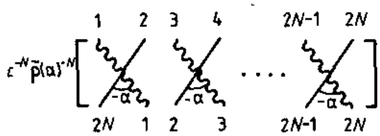
fulfil the usual equation

$$[\tilde{\tau}^{(\mathrm{alt})}(\theta,\alpha),\tilde{\tau}^{(\mathrm{alt})}(\theta',\alpha)]=0. \tag{2.22}$$

Let us now investigate the properties of $\tilde{\tau}^{(\mathrm{alt})}_{(\theta)}$. First, for $\theta=0$ we find

$\tilde{\tau}^{(\mathrm{alt})}(0,\alpha)=c^N$  (2.23)

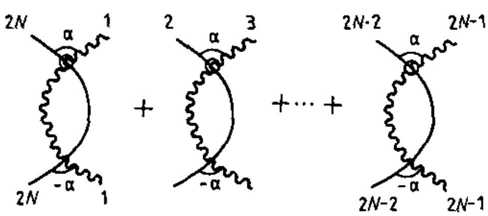
This is not a shift operator like (2.9), but rather looks like a light-cone transfer matrix [9]. The inverse operator $[\tilde{\tau}^{(\mathrm{alt})}(\theta)]^{-1}$ is given by

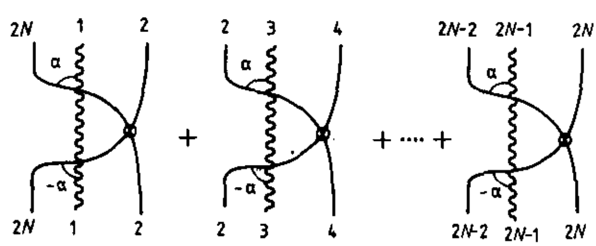
$\tilde{\tau}^{-1}(0,\alpha)=c^{-N}\tilde{\rho}(\alpha)^{-N}$  (2.24)

where we have used equation (2.17). We are now in a position to compute the logarithmic derivative of $\tilde{\tau}^{(\mathrm{alt})}(\theta)$ at $\theta=0$. We find

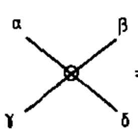
$$\tilde{N}_{(\alpha)}\tilde{H}=\frac{\partial}{\partial\theta}\log\tilde{\tau}^{(\mathrm{alt})}(\theta,\alpha)|_{\theta=0}=\tilde{N}_{(\alpha)}(H_2+H_3) \tag{2.25}$$

where

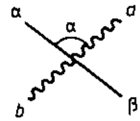
$\tilde{\rho}(0)=H_2=$  (2.26)

$c\tilde{\rho}(0)H_3=$  (2.27)

and $\tilde{N}_{(\alpha)}$ is a normalization that we shall choose later to our convenience. Here

 $=\dot{R}^{\alpha\beta}_{\gamma\delta}(0)$

and



$$= [\dot{t}_{\alpha\beta}(\alpha)]_{ab}. \tag{2.28}$$

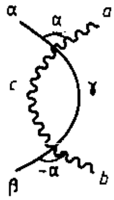
In H_2 (H_3) we collected the terms originated when $\partial/\partial\theta$ acted on an operator $\tilde{t}^{(2k+1)}(\theta) \times (t^{(2k)}(\theta))$. H_2 contains nearest neighbour interactions (two sites) whereas in H_3 there are next-to-nearest neighbour couplings. The three-site couplings come from the $\tilde{t}^{(2k+1)}(0)$ which does not decouple as the $t^{(2k)}(\theta)|_{\theta=0}$ does (equation (2.10).)

Let us write H_2 and H_3 in analytic form

$$H_2(\alpha) = \sum_{M=1}^{N} h_{2M,2M+1}(\alpha) \qquad H_3(\alpha) = \sum_{M=1}^{N} h_{2M,2M+1,2M+2}(\alpha) \tag{2.29}$$

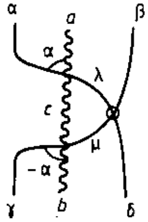
Here $2N+1 \equiv 1$, $2N+2 \equiv 2$

$$h_{\alpha a,\beta b}(\alpha) = \sum_{c\gamma} [\dot{t}(\alpha)_{\alpha\gamma}]_{ac} [\tilde{t}_{\gamma\beta}(-\alpha)]_{cb} \frac{\tilde{N}(\alpha)}{\rho(\alpha)} = \tag{2.30}$$



$$h_{\alpha a\beta,\gamma b\delta}(\alpha) = \frac{\tilde{N}_{(\alpha)}}{c\rho(\alpha)} \sum_{c\lambda\mu} [\tilde{t}_{\alpha\lambda}(\alpha)]_{ac} [\dot{t}_{\lambda\delta}(0)]_{\beta\mu} [\tilde{t}_{\mu\gamma}(-\alpha)]_{cb}$$

$$= \tag{2.31}$$



In conclusion, we have just constructed a one-parameter family

$$\tilde{H}(\alpha) = H_2(\alpha) + H_3(\alpha) \tag{2.32}$$

of integrable Hamiltonians from a YB solution. Besides α, this Hamiltonian may depend on one (γ) or two (γ and k) continuous parameters. The latter case corresponds to elliptic YB solutions.

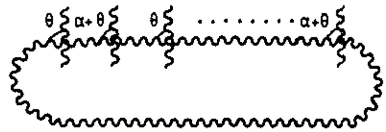
We say that (2.32) is an integrable Hamiltonian, because it commutes with the one-parameter family of transfer matrices $\tilde{\tau}^{(\mathrm{alt})}(\theta, \alpha)$:

$$[H_2(\alpha) + H_3(\alpha), \tilde{\tau}^{(\mathrm{alt})}(\theta, \alpha)] = 0 \qquad \forall\theta. \tag{2.33}$$

Let us now introduce the momentum operator appropriate for the alternating configuration (2.19). In the usual case (2.2) the transfer matrix at $\theta = 0$ gives the one-step shift operator (2.9) and the momentum is just its logarithm. In our alternating case (2.19) the basic object will be a two-step shift operator since a one-step shift would exchange the nature of the sites. We can relate this two-step shift with transfer matrices as follows. Let us consider the transfer matrix $\tilde{\tau}^{(\mathrm{alt})}(\theta, \alpha)$. It follows from

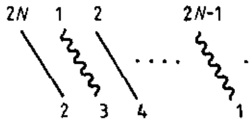
$\tilde{\tau}^{(\mathrm{alt})}(\theta, \alpha)$ (equation (2.21)) by exchanging the lines ______ and ∿∿∿∿. That is,

$$\bar{\tau}^{(\mathrm{alt})}(\theta, \alpha) =$$



Using (2.17) we find after a little graphical calculation

$$\bar{\tau}^{(\mathrm{alt})}(\theta, -\beta)\tilde{\tau}^{\mathrm{alt}}(\theta, \beta)|_{\theta=0} = \tag{2.34}$$



Therefore, we define the momentum as

$$P \equiv \mathrm{i} \ln[\bar{\tau}^{(\mathrm{alt})}(0, -\alpha)\tilde{\tau}^{(\mathrm{alt})}0, \alpha)]. \tag{2.35}$$

The family $\bar{\tau}^{(\mathrm{alt})}(\theta, \beta)$ generates also a commuting family

$$\bar{\tau}^{(\mathrm{alt})}(\theta, \beta), \bar{\tau}^{(\mathrm{alt})}(\theta', \beta)] = 0 \tag{2.36}$$

when $\beta = -\alpha$ this family commutes with the $\tilde{\tau}^{(\mathrm{alt})}(\theta, \alpha)$

$$[\bar{\tau}^{(\mathrm{alt})}(\theta, -\alpha), \tilde{\tau}^{(\mathrm{alt})}(\theta', \alpha)] = 0. \tag{2.37}$$

This is a consequence of the YB equation

$$\begin{aligned}[\tilde{t}_{\gamma\alpha}(\theta - \theta' + \alpha)]_{ca} \tilde{T}^{(\mathrm{alt})}_{\alpha\beta}(\theta, \alpha) \bar{T}^{(\mathrm{alt})}_{ab}(\theta', -\alpha) \\ = \bar{T}^{(\mathrm{alt})}_{cd}(\theta', -\alpha) \tilde{T}^{(\mathrm{alt})}_{\gamma\delta}(\theta, \alpha)[\tilde{t}_{\delta\beta}(\theta - \theta' + \alpha)]_{db}\end{aligned} \tag{2.38}$$

$\bar{\tau}^{(\mathrm{alt})}(\theta, \alpha)$ generates operators commuting both with $\tilde{\tau}^{(\mathrm{alt})}(\theta, \alpha)$ and $\tilde{H}$. The first one

$$\bar{H} = \bar{N}(\alpha) \frac{\partial}{\partial \theta} \ln \bar{\tau}^{(\mathrm{alt})}(\theta, -\alpha)|_{\theta=0} \tag{2.39}$$

(with $N_{(\alpha)}$ being an appropriate normalization constant) contains two and three-site couplings just as $\tilde{H}$ (equations (2.26) and (2.27)] but with the lines ______ and ∿∿∿∿ exchanged.

3. A spin-$\frac{1}{2}$–spin-1 anisotropic integrable Hamiltonian

We apply in this section the general framework presented in section 2 to the specific case of the six-vertex model and the YB solution obtained by fusing it. That is, we take

$$[t_{\alpha\beta}(\theta)]_{\gamma\delta} = S^{\beta\delta}_{\alpha\gamma}(\theta)$$

with

$$S(\theta) = \begin{pmatrix} \sinh(\theta + \gamma) & 0 & 0 & 0 \\ 0 & \sinh\theta & \sinh\gamma & 0 \\ 0 & \sinh\gamma & \sinh\theta & 0 \\ 0 & 0 & 0 & \sinh(\theta + \gamma) \end{pmatrix} \tag{3.1}$$

$\alpha, \beta, \gamma, \delta = \pm\frac{1}{2}$ and

$$[\tilde{t}_{ab}(\theta)]_{\alpha\beta} = [\tilde{t}_{\beta\alpha}(\theta)]_{ba} = \tilde{S}^{a\alpha}_{b\beta}(\theta)$$

a, $b=0, \pm 1$, with [2]

$$\tilde{S}(\theta)=\begin{pmatrix} A(\theta) & 0 & 0 & 0 & 0 & 0 \\ 0 & B_-(\theta) & C(\theta) & 0 & 0 & 0 \\ 0 & C(\theta) & B_+(\theta) & 0 & 0 & 0 \\ 0 & 0 & 0 & B_+(\theta) & C(\theta) & 0 \\ 0 & 0 & 0 & C(\theta) & B_-(\theta) & 0 \\ 0 & 0 & 0 & 0 & 0 & A(\theta) \end{pmatrix} \tag{3.2}$$

and

$$A(\theta)=\sinh(\theta+\tfrac{3}{2}\gamma) \qquad B_\pm(\theta)=\sinh\left(\theta\pm\frac{\gamma}{2}\right) \qquad C(\theta)=\sinh\gamma\sqrt{2\cosh\gamma} \tag{3.3}$$

This is a regular YB solution. Equations (2.8) and (2.17) hold with

$$\rho(\theta)=\tfrac{1}{2}(\cosh 2\gamma-\cosh 2\theta) \qquad \tilde{\rho}(\theta)=\tfrac{1}{2}(\cosh 3\gamma-\cosh 2\theta). \tag{3.4}$$

Inserting (3.1)-(3.3) in (2.30), (2.31) yields the matrix elements of our integrable Hamiltonian. We choose for simplicity

$$\tilde{N}(\alpha)=\sinh\gamma(\cosh 2\alpha-\cosh 3\gamma). \tag{3.5}$$

The Hamiltonian thereby obtained is invariant under rotations around the z axis and also under reflections on the xy plane. Due to these symmetries

$$\begin{aligned} &h_{\alpha a\beta,\gamma b\delta}=0 \qquad \text{if } \alpha+a+\beta\neq\gamma+b+\delta \\ &h_{\alpha a,\beta b}=0 \qquad \text{if } \alpha+a\neq\beta+b \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} &h_{\alpha a\beta,\gamma b\delta}=h_{-\alpha-a-\beta,-\gamma-b-\delta} \\ &h_{\alpha a,\beta b}=h_{-\alpha-a,-\beta-b}. \end{aligned} \tag{3.7}$$

In addition the Hamiltonian $\tilde{H}(\alpha)$ is a symmetric matrix.

We find for the non-vanishing matrix elements of $H_3(\alpha)$:

$$\begin{aligned} &h^{1/2\,1\,1/2}_{1/2\,1\,1/2}=\cosh\gamma(\cosh 2\alpha-\cosh 3\gamma) \\ &h^{-1/2\,1\,-1/2}_{-1/2\,1\,-1/2}=h^{1/2\,0\,1/2}_{1/2\,0\,1/2}=\cosh\gamma h^{1/2\,0\,-1/2}_{-1/2\,0\,1/2}=\cosh\gamma(\cosh 2\alpha-\cosh\gamma) \\ &h^{-1/2\,1\,1/2}_{-1/2\,1\,1/2}=h^{1/2\,0\,-1/2}_{1/2\,0\,-1/2}=-\sinh^2 2\gamma \\ &h^{-1/2\,1\,-1/2}_{1/2\,-1\,+1/2}=-\sinh\gamma\sinh 2\gamma \\ &h^{1/2\,0\,1/2}_{-1/2\,1\,1/2}=-\sinh 2\,\gamma\sqrt{2\cosh\gamma}\sinh(\alpha+\gamma/2) \\ &h^{-1/2\,0\,1/2}_{-1/2\,1\,-1/2}=2\sinh\gamma\sqrt{2\cosh\gamma}\sinh(\alpha-3\gamma/2)=h^{1/2\,0\,1/2}_{1/2\,1\,-1/2} \\ &h^{1/2\,0\,-1/2}_{-1/2\,1\,-1/2}=-\sinh(\alpha-\gamma/2)\sqrt{2\cosh\gamma}\sinh 2\gamma \\ &h^{-1/2\,1\,1/2}_{1/2\,1\,-1/2}=h^{1/2\,1\,-1/2}_{-1/2\,1\,1/2}=\cosh(2\alpha+2\gamma)-\cosh\gamma. \end{aligned} \tag{3.8}$$

For $H_2(\alpha)$ we find after neglecting a trivial term proportional to the identity operator

$$h_{1/2\,1}^{1/2\,1}=-\sinh\gamma(2\cosh 2\gamma+1) \qquad h_{1/2,0}^{-1/2,1}=-2\sinh\gamma\sqrt{2\cosh\gamma}\cosh(\alpha+\gamma/2)$$
$$h_{-1/2\,1}^{-1/2,1}=-h_{1/2\,0}^{1/2\,0}=\sinh\gamma. \tag{3.9}$$

These operators can be conveniently written in terms of spin-$\frac{1}{2}$ and spin-1 operators. We find for the three-site operator $H_3(\alpha)$

$$\begin{aligned}
h_{2n,2n+1,2n+2}(\alpha)
&=\tfrac{1}{2}(\cosh 2\alpha-\cosh\gamma)\sigma+\sinh\gamma\sinh(2\alpha+\gamma)\sigma(S_z)^2)^2+\sinh 2\gamma\sqrt{2\cosh\gamma}\\
&\quad\times\left[\frac{\sinh(\alpha-3\gamma/2)}{\cosh\gamma)}U'-\sinh\alpha\cosh\frac{\gamma}{2}U-\cosh\alpha\sinh\frac{\gamma}{2}V\right]\\
&\quad-\tfrac{1}{2}\sinh\gamma\sinh 2\gamma W+\frac{\cosh\gamma}{2}(\cosh 3\gamma+\cosh 2\alpha-2\cosh\gamma)\\
&\quad\times(\sigma_z\otimes 1\otimes\sigma_z)-\frac{\sinh^2 2\gamma}{2}[\sigma_z\otimes S_z\otimes 1+1\otimes S_z^2\otimes 1]\\
&\quad+\frac{\cosh\gamma}{2}(\cosh 2\alpha-\cosh 3\gamma)\mathbb{1}
\end{aligned}\tag{3.10}$$

where
$$\begin{aligned}
\sigma&\equiv\sigma_x\otimes 1\otimes\sigma_x'+\sigma_y\otimes 1\otimes\sigma_y'\\
U&\equiv\sigma_x\otimes S_x\otimes 1+\sigma_y\otimes S_y\otimes 1\\
U'&=1\otimes S_x\otimes\sigma_x'+1\otimes S_y\otimes\sigma_y'\\
V&\equiv\sigma_x\otimes\{S_x,S_z\}\otimes\sigma_z'+\sigma_y\otimes\{S_y,S_z\}\otimes\sigma_z'\\
W&\equiv\sigma_-\otimes(S_+)^2\otimes\sigma_-'+\sigma_+\otimes(S_-)^2\otimes\sigma_+'
\end{aligned}\tag{3.11}$$

σ_a and σ_a' are Pauli matrices acting on sites $2n$ and $2n+2$, respectively. The spin-1 operators

$$S_x=\frac{1}{2}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\qquad S_y=\frac{i}{2}\begin{pmatrix}0&-1&0\\1&0&-1\\0&1&0\end{pmatrix}\qquad S_z=\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}\tag{3.12}$$

act on site $2n+1$.

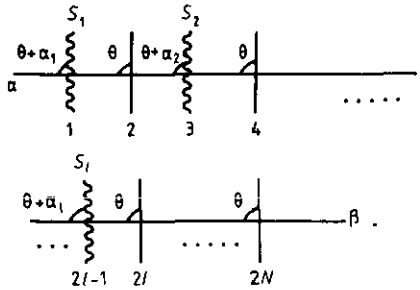
We find for the two-site part $H_2(\alpha)$

$$h_{2n,2n+1}(\alpha)=-2\left[\cosh^2\gamma\sigma_z S_z+\sqrt{\cosh\gamma}\cosh\left(\alpha+\frac{\gamma}{2}\right)U+\sinh^2\gamma(S_z)^2\right].\tag{3.13}$$

We have considered up to now the non-zero gap regime where both the YB solutions $\bar{t}(\theta)$, $\tilde{t}(\theta)$ and the Hamiltonian $\tilde{H}(\alpha)$ are hyperbolic functions of θ, γ and α. By changing $\alpha\to\mathrm{i}\alpha$, $\gamma\to\mathrm{i}\gamma$, $\theta\to\mathrm{i}\theta$, we obtain the gapless regime.

The Hamiltonian $\tilde{H}(\alpha)$ (equations (3.10)–(3.13)) is the simplest one combining two types of spin. It is straightforward to generalize it. For example, using $\bar{R}(\theta)$ and $\tilde{t}(\theta)$, where $\bar{R}(\theta)$ (see (2.14)) is the spin-1 anisotropic R-matrix [2], we would get in this way a Hamiltonian $\bar{H}(\alpha)$ coupling two spins-1 with one spin-$\frac{1}{2}$ instead of two spins-$\frac{1}{2}$ and one spin-1 as is the case for $\tilde{H}(\alpha)$. More generally one can take any type of higher spin and dispose them in any definite order. That is, we can consider as a generalized alternating YB operator

$T^{(\mathrm{gen})}_{\alpha\beta}(\theta, \boldsymbol{\alpha}) =$



Here the full line stands for a spin S multiplet ($-S \leqslant \alpha, \beta \leqslant +S$) and the vertical line at site $(2l-1)$ corresponds to a spin S_l multiplet ($-S_l \leqslant \alpha_l, \beta_l \leqslant +S_l$).

To be general, we consider a site-dependent parameter α_l. We have then a class of integrable Hamiltonians that follows from the transfer matrix

$$\tau^{(\mathrm{gen})}(\theta, \boldsymbol{\alpha}) = \sum_{\alpha=-s}^{s} T_{\alpha\alpha}(\theta, \boldsymbol{\alpha}) \tag{3.14}$$

as

$$H(\underset{\sim}{\alpha}) = N(\underset{\sim}{\alpha}) \frac{\partial}{\partial\theta} \ln \tau^{(\mathrm{gen})}(\theta, \underset{\sim}{\alpha})|_{\theta=0}. \tag{3.15}$$

These Hamiltonians have a structure similar to (2.26), (2.27). That is, the H_3 part couples the spin S_l at site $(2l-1)$ with its neighbouring spins S at sites $2l-2$ and $2l$. The H_2 provides a coupling between S_l (at $2l-1$) and S at site $2l-2$. The explicit matrix elements of this general class of Hamiltonians can be obtained through a straightforward and long calculation from the YB solutions obtained by fusion in [3, 6].

We have considered here generically anisotropic Hamiltonians. As is clear, their isotropic limits, as

$$\tilde{H}_{iso} \equiv \lim_{\gamma\to 0} \frac{1}{\gamma^2} [H_2(\alpha = \gamma a) + H_3(\alpha = \gamma a)]$$

are also integrable.

4. Bethe ansatz solution of the models with different types of spins

In this section we solve the spin-$\frac{1}{2}$-spin-1 models introduced in section 3 (equations (2.29)-(2.31) and (2.39)). We shall call $N_{1/2}$ and N_1 the numbers of sites occupied by spin-$\frac{1}{2}$ and spin-1 atoms, respectively. In (2.19)—since these sites are alternating—$N_{1/2} = N_1 = N_1$.

In order to find the eigenvectors and eigenvalues of $\tilde{H}(\alpha)$, we construct those of $\tilde{\tau}^{(\mathrm{alt})}(\theta, \alpha)$ by algebraic Bethe ansatz. This can be done in analogy to the construction in [6] for pure (non-alternating) spin S models. We find for the eigenvalues of the $\tilde{\tau}^{(\mathrm{alt})}(\theta, \alpha)$ in the trigonometric regime

$$\tilde{\Lambda}(\theta, \alpha) = \Lambda_+(\theta, \alpha) + \Lambda_-(\theta, \alpha) \tag{4.1}$$

$$\Lambda_+(\theta, \alpha) = [\sin(\gamma+\theta)]^{N_{1/2}}[\sin(3\gamma/2+\alpha+\theta)]^{N_1} \prod_j^r \frac{\sinh(\lambda_j + \mathrm{i}\gamma/2 - \mathrm{i}\theta)}{\sinh(\lambda_j - \mathrm{i}\gamma/2 - \mathrm{i}\theta)} \tag{4.2}$$

$$\Lambda_{-}(\theta,\alpha)=[\sin\theta]^{N_{1/2}}[\sin(\alpha+\theta-\gamma/2)]^{N_1}\prod_j^r\frac{\sinh(\lambda_j-\mathrm{i}3\gamma/2-\mathrm{i}\theta)}{\sinh(\lambda_j-\mathrm{i}\gamma/2-\mathrm{i}\theta)}. \tag{4.3}$$

The λ_j $(1\leq j\leq r)$ are solutions of the Bethe ansatz equations

$$\left[\frac{\sinh(\lambda_j+\mathrm{i}\gamma/2)}{\sinh(\lambda_j-\mathrm{i}\gamma/2)}\right]^{N_{1/2}}\left[\frac{\sinh(\lambda_j+\mathrm{i}(\gamma+\alpha))}{\sinh(\lambda_j-\mathrm{i}(\gamma-\alpha))}\right]^{N_1}=\prod_{k=1}^r\frac{\sinh(\lambda_j-\lambda_k+\mathrm{i}\gamma)}{\sinh(\lambda_j-\lambda_k-\mathrm{i}\gamma)}. \tag{4.4}$$

These equations reduce to the well known pure spin-$\frac{1}{2}$ and spin-1 equations when $N_1=0$, $N_{1/2}=N$ and $N_{1/2}=0$, $N_1=N$, respectively (in this latter case α can be transformed out of (4.4)). According to (2.25)

$$\tilde{E}(\alpha)=\tilde{N}_\alpha\frac{\partial}{\partial\theta}\ln\tilde{\Lambda}(\theta,\alpha)|_{\theta=0} \tag{4.5}$$

where the choice of $\tilde{N}_\alpha$ corresponding to (3.5) is

$$\tilde{N}_\alpha=(\cosh 2\alpha-\cosh 3\gamma)\sinh\gamma. \tag{4.6}$$

This, through (4.1)-(4.3) yields

$$\tilde{E}(\alpha)=\tilde{N}_\alpha[N_{1/2}\cot\gamma+N_1\cot(3\gamma/2+\alpha)-\sum_{j=1}^r\phi'(\lambda_j,\gamma/2)]. \tag{4.7}$$

Here we have used the notation

$$\phi(\lambda,\alpha)=\mathrm{i}\ln\frac{\sinh(\lambda+\mathrm{i}\alpha)}{\sinh(\lambda-\mathrm{i}\alpha)} \tag{4.8}$$

and ϕ' is the derivative of ϕ with respect to λ.

The momentum of a solution of (4.4) can be calculated using the definition (2.35). After a derivation analogus to that of (4.2) we find for the eigenvalue $\bar{\tau}^{(\mathrm{alt})}(\theta,-\alpha)$

$$\bar{\Lambda}(\theta,\alpha)=\sum_{s=-1}^{+1}\Lambda_s(\theta,\alpha) \tag{4.9}$$

with

$$\Lambda_{+1}(\theta,\alpha)=[\sin(\theta+\gamma)\sin(\theta+2\gamma)]^{N_1}[\sin(\theta-\alpha-\gamma/2)]^{N_{1/2}}\prod_j^r\frac{\sinh(\lambda_j-\mathrm{i}\theta+\mathrm{i}\gamma)}{\sinh(\lambda_j-\mathrm{i}\theta-\mathrm{i}\gamma)} \tag{4.10}$$

where the λ_j satisfy the Bethe ansatz equations (4.4). Since Λ_0 and Λ_{-1}, as well as their θ-derivatives are zero at $\theta=0$, only Λ_I is relevant, and (2.35) yields

$$P=\sum_{l=1}^r[\phi(\lambda_l,\gamma/2)+\phi(\lambda_l,\gamma)]. \tag{4.13}$$

Here we have subtracted a constant in order to make the momentum of the ferromagnetic vacuum (all spins up to $r=0$) vanishing.

Before solving the (4.4) Bethe ansatz equations, let us give the eigenvalues of the Hamiltonian $\bar{H}(\alpha)$. Due to (2.39) and (4.9), (4.10)

$$\bar{E}(\alpha)=\bar{N}_\alpha\frac{\partial}{\partial\theta}\ln\bar{\Lambda}(\theta,\alpha)|_{\theta=0}=\bar{N}_\alpha\frac{\partial}{\partial\theta}\ln\Lambda_{+1}(\theta,\alpha)|_{\theta=0} \tag{4.14}$$

which yields

$$\bar{E}=\bar{N}_\alpha\left[N_{1/2}(\cot\gamma+\cot 2\gamma)+N_1\cot(\alpha+\gamma/2)-\sum_{j=1}^{r}\phi'(\lambda_j,\gamma)\right] \tag{4.15}$$

with $\bar{N}_\alpha$ being a normalization factor.

Notice, that the eigenvectors of $\bar{H}(\alpha)$ are the same as those of $\tilde{H}(\alpha)$. This is because

$$[\tilde{H}(\alpha),\bar{H}(\alpha)]=0 \tag{4.16}$$

which is a consequence of (2.37).

5. Solution of the Bethe ansatz equations

In the following we give the solution of the (4.4) Bethe ansatz equations. For the sake of simplicity we deal with the $\alpha=0$ case only, and we suppose that $\gamma<\pi/4$.

The ground state is formed by real roots η_α (like the ground state of a spin-$\frac{1}{2}$ chain) and pairs of complex conjugate roots with real parts ξ_α. In the $N\to\infty$ limit these latter roots become 2-strings

$$\xi_\alpha\pm i\gamma/2 \tag{5.1}$$

(like the roots in the case of the spin-1 chain). Substituting $\lambda_j=\eta_\alpha$ and $\lambda_j=\xi_\alpha\pm i\gamma/2$, and taking the logarithm we obtain

$$\begin{aligned}N_{1/2}\phi(\eta_\alpha,\gamma/2)&+N_1\phi(\eta_\alpha,\gamma)\\&=2\pi J_\alpha+\sum_\beta\phi(\eta_\alpha-\eta_\beta,\gamma)\\&\quad+\sum_\beta(\phi(\eta_\alpha-\xi_\beta,\gamma/2)+\phi(\eta_\alpha-\xi_\beta,3\gamma/2))\end{aligned} \tag{5.2}$$

and

$$\begin{aligned}N_{1/2}\phi(\xi_\alpha,\gamma)&+N_1(\phi(\xi_\alpha,\gamma/2)+\phi(\xi_\alpha,3\gamma/2))\\&=2\pi I_\alpha+\sum_\beta(\phi(\xi_\alpha-\eta_\beta,\gamma/2)+\phi(\xi_\alpha-\eta_\beta,3\gamma/2))\\&\quad+\sum_\beta(2\phi(\xi_\alpha-\xi_\beta,\gamma)+\phi(\xi_\alpha-\xi_\beta,2\gamma)).\end{aligned} \tag{5.3}$$

Here J_α and $I_\alpha\in Z+\frac{1}{2}$, ϕ is given by (4.8), and we have used the properties

$$\begin{aligned}&\phi(\lambda-i\gamma/2,\gamma)+\phi(\lambda+i\gamma/2,\gamma)=\phi(\lambda,\gamma/2)+\phi(\lambda,3\gamma/2)\\&\phi(\lambda-i\gamma/2,\gamma/2)+\phi(\lambda+i\gamma/2,\gamma/2)=\phi(\lambda,\gamma)(\mathrm{mod}\,2\pi).\end{aligned} \tag{5.4}$$

If we choose the cut of the ln function in (4.8) so, that ϕ is continuous for real λ, in the ground-state the J_α and the I_α form monotonous sequences

$$\begin{aligned}&J_{\alpha+1}-J_\alpha=1\\&I_{\alpha+1}-I_\alpha=1.\end{aligned} \tag{5.5}$$

The numbers of η_α and ξ_α are $N_{1/2}/2$ and $N_1/2$, respectively. In the thermodynamic limit the spacing between the neighbouring ηs and ξs tends to zero as N^{-1}. We define the root-densities as

$$\rho_{1/2}(\eta_\alpha)=\lim_{N_{1/2}\to\infty}\frac{1}{N_{1/2}(\eta_{\alpha+1}-\eta_\alpha)}$$
$$\rho_1(\xi_\alpha)=\lim_{N_1\to\infty}\frac{1}{N_1(\xi_{\alpha+1}-\xi_\alpha)}. \tag{5.6}$$

The equations determining these quantities can be obtained from (5.2) and (5.3) by standard methods:

$$\begin{aligned}N_{1/2}\phi'(\eta,\gamma/2)+N_1\phi'(\eta,\gamma)\\ &=N_{1/2}2\pi\rho_{1/2}(\eta)+N_{1/2}\int_{-\infty}^{\infty}\phi'(\eta-\eta',\gamma)\rho_{1/2}(\eta')\,\mathrm{d}\eta'\\ &\quad+N_1\int_{-\infty}^{\infty}(\phi'(\eta-\xi,\gamma/2)+\phi'(\eta-\xi,3\gamma/2))\rho_1(\xi)\,\mathrm{d}\xi\end{aligned} \tag{5.7}$$

and

$$\begin{aligned}N_{1/2}\phi'(\xi,\gamma)+N_1(\phi'(\xi,\gamma/2)+\phi'(\xi,3\gamma/2))\\ &=N_1 2\pi\rho_1(\xi)+N_{1/2}\int_{-\infty}^{\infty}(\phi'(\xi-\eta,\gamma/2)+\phi'(\xi-\eta,3\gamma/2))\rho_{1/2}(\eta)\,\mathrm{d}\eta\\ &\quad+N_1\int_{-\infty}^{\infty}(2\phi'(\xi-\xi',\gamma)+\phi'(\xi-\xi',2\gamma))\rho_1(\xi')\,\mathrm{d}\xi'.\end{aligned} \tag{5.8}$$

These equations solved by Fourier transformation yield

$$\rho^0_{1/2}(\eta)=\frac{1}{2\gamma\cosh(\pi\eta/\gamma)}$$
$$\rho^0_1(\xi)=\frac{1}{2\gamma\cosh(\pi\xi/\gamma)}. \tag{5.9}$$

Excitations can be introduced by leaving holes in the η and ξ distributions and introducing complex λs not forming 2-strings. The construction of such solutions of (4.4) closely parallels the analysis of the Bethe ansatz equations of the pure spin-$\frac{1}{2}$ model [8]. Here we give the main point only. The holes can be introduced by leaving holes in the J_α and I_α sequences

$$J_{\alpha+1}-J_\alpha=1+\delta_{\alpha,\alpha_h}$$
$$I_{\beta+1}-I_\beta=1+\delta_{\beta,\beta_h} \tag{5.10}$$

where $\delta_{\alpha,\alpha'}$ is the Kronecker-symbol. In this case the root-densities $\rho_{1/2}$ and ρ_1 can be defined as

$$\rho_{1/2}(\eta_\alpha)+\frac{1}{N_{1/2}}\delta(\eta_\alpha-\eta_h)\equiv\sigma_{1/2}(\eta_\alpha)=\lim_{N_{1/2}\to\infty}\frac{J_{\alpha+1}-J_\alpha}{N_{1/2}(\eta_{\alpha+1}-\eta_\alpha)}$$
$$\rho_1(\xi_\beta)+\frac{1}{N_1}\delta(\xi_\beta-\xi_h)\equiv\sigma_1(\xi_\beta)=\lim_{N_1\to\infty}\frac{I_{\beta+1}-I_\beta}{N_1(\xi_{\beta+1}-\xi_\beta)}. \tag{5.12}$$

Here $\delta(x)$ is the Dirac δ-function, and the σs are continuous functions giving the densities of the *roots* and *holes*. Keeping the positions of the holes and the complex λs as parameters we have the equations

$$N_{1/2}\phi'(\eta,\gamma/2)+N_1\phi'(\eta,\gamma)$$
$$=N_{1/2}2\pi\sigma_{1/2}(\eta)+N_{1/2}\int \mathrm{d}\eta\,\phi'(\eta-\eta',\gamma)\sigma_{1/2}(\eta')-\sum_h \phi'(\eta-\eta_h,\gamma)$$
$$+N_1\int(\phi'(\eta-\xi,\gamma/2)+\phi'(\eta-\xi,3\gamma/2))\sigma_1(\xi)\,\mathrm{d}\xi$$
$$-\sum_h(\phi'(\eta-\xi_h,\gamma/2)+\phi'(\eta-\xi_h,3\gamma/2))$$
$$+\sum_n \phi'(\eta-\Lambda_n,\gamma) \tag{5.13}$$

and

$$N_{1/2}\phi'(\xi,\gamma)+N_1(\phi'(\xi,\gamma/2)+\phi'(\xi,3\gamma/2))$$
$$=N_1 2\pi\sigma_1(\xi)+N_{1/2}\int(\phi'(\xi-\eta,\gamma/2)+\phi'(\xi-\eta,3\gamma/2))\sigma_{1/2}(\eta)\,\mathrm{d}\eta$$
$$-\sum_h(\phi'(\xi-\eta_h,\gamma/2)+\phi'(\xi-\eta_h,3\gamma/2))$$
$$+N_1\int(2\phi'(\xi-\xi',\gamma)+\phi'(\xi-\xi',2\gamma))\sigma_1(\xi')\,\mathrm{d}\xi'$$
$$-\sum_h(2\phi'(\xi-\xi_h,\gamma)+\phi'(\xi-\xi_h,2\gamma))$$
$$+\sum_n(\phi'(\xi-\Lambda_n,\gamma/2)+\phi'(\xi-\Lambda_n,3\gamma/2)). \tag{5.14}$$

Here the Λ_n are the complex rapidities and they are determined by the equations

$$N_{1/2}\phi(\Lambda_n,\gamma/2)+N_1\phi(\Lambda_n,\gamma)$$
$$=2\pi J_n+\sum_{n'}\phi(\Lambda_n-\Lambda_{n'})+N_{1/2}\int \mathrm{d}\eta\,\phi(\Lambda_n-\eta,\gamma)\sigma_{1/2}(\eta)$$
$$-\sum_h \phi(\Lambda_n-\eta_h,\gamma/2)$$
$$+N_1\int(\phi(\Lambda_n-\xi,\gamma/2)+\phi(\Lambda_n-\xi,3\gamma/2))\sigma_1(\xi)\,\mathrm{d}\xi$$
$$-\sum_h(\phi(\Lambda_n-\xi_h,\gamma/2)+\phi(\Lambda_n-\xi_h,3\gamma/2)). \tag{5.15}$$

The equations (5.13) and (5.14) can be solved, and then $\sigma_{1/2}$ and σ_1 can be eliminated from (5.15). This way we arrive at a system of equations which contains η_h, ξ_h and Λ_n only. The analytic properties of the equations in (5.15) are different depending on if Λ_n is a 'very close' ($|\mathrm{Im}\,\Lambda_n|<\gamma/2$), a 'close' ($\gamma/2<|\mathrm{Im}\,\Lambda_n|<\gamma$), a 'wide' ($\gamma<|\mathrm{Im}\,\Lambda_n|<3\gamma/2$) or a 'very wide' ($3\gamma/2<|\mathrm{Im}\,\Lambda_n|(<\pi/2)$) root. Actually it turns out, that (5.15) can be satisfied only if the very close, close and wide roots come in *trios* so that the members of such a trio have common real parts, and the spacings in the imaginary direction are $\mathrm{i}\gamma$:

$$\Lambda_n=\chi_n \qquad \Lambda_n^{\pm}=\chi_n\pm\mathrm{i}\gamma \qquad |\mathrm{Im}\,\chi_n|<\gamma/2. \tag{5.16}$$

(These are generalized 3-strings.) There are no such restrictions on the very wide roots which we write in the form

$$\Lambda_n = \chi_n + \mathrm{i}\,\mathrm{sgn}(\mathrm{Im}\,\chi_n)\gamma \qquad |\mathrm{Im}\,\chi_n| > \gamma/2. \tag{5.17}$$

The equations for the positions of the holes can be reconstructed from the σs. Finally the densities, together with the higher level Bethe ansatz equations (which give the positions of the holes and χ_n) can be given:

$$\sigma_{1/2}(\eta) = \rho^0_{1/2}(\eta) + \frac{1}{N_{1/2}}\sum_h \frac{1}{2\gamma\cosh(\pi(\eta-\xi_h)/\gamma)} \tag{5.18}$$

$$\sigma_1(\xi) = \rho^0_1(\xi) + \frac{1}{N_1}\sum_h \frac{1}{2\gamma\cosh(\pi(\xi-\eta_h)/\gamma)} + \frac{1}{N_1}\sum_h \psi'(\xi-\xi_h) - \frac{1}{N_1}\frac{\pi}{\pi-2\gamma}\sum_m \phi'\left(\frac{(\xi-\chi_n)\pi}{\pi-2\gamma}, \frac{(\gamma/2)\pi}{\pi-2\gamma}\right) \tag{5.19}$$

$$N_{1/2} 2\tan^{-1}\left(\tanh\frac{\pi\eta_h}{2\gamma}\right) = 2\pi J_h - \sum_{h'} 2\tan^{-1}\left(\tanh\frac{\pi(\eta_h-\xi_{h'})}{2\gamma}\right) \tag{5.20}$$

$$N_1 2\tan^{-1}\left(\tanh\frac{\pi\xi_h}{2\gamma}\right) = 2\pi I_h - \sum_{h'} 2\tan^{-1}\left(\tanh\frac{\pi(\xi_h-\eta_{h'})}{2\gamma}\right) - \sum_{h'}\psi(\xi_h-\xi_{h'}) + \sum_n \phi\left(\frac{(\xi_h-\chi_n)\pi}{\pi-2\gamma}, \frac{(\gamma/2)\pi}{\pi-2\gamma}\right) \tag{5.21}$$

and

$$\sum_h \phi\left(\frac{(\chi_n-\xi_h)\pi}{\pi-2\gamma}, \frac{(\gamma/2)\pi}{\pi-2\gamma}\right) = 2\pi I_n + \sum_{n'} \phi\left(\frac{(\chi_n-\chi_{n'})\pi}{\pi-2\gamma}, \frac{\gamma\pi}{\pi-2\gamma}\right) \tag{5.22}$$

with

$$\psi(x) = \int_{-\infty}^{\infty} \phi\left(\frac{(x-y)\pi}{\pi-2\gamma}, \frac{(\gamma/2)\pi}{\pi-2\gamma}\right)\frac{1}{2\gamma\cosh(\pi y/\gamma)}\,\mathrm{d}y. \tag{5.23}$$

It is interesting, that in the equations of η_h the other $\eta_{h'}$ and χ_n do not appear, and that the χ_n are directly related to the ξ_h only. It is interesting to note also, that for an $S^z=0$ state (the total number of λ_s is $N_{1/2}/2+N_1$) the number of χ_n is half the number of ξ_h. (If $S^z\neq 0$, the relation between the numbers of the η_h, ξ_h and σ_n is more complicated, and also terms decaying as $N\to\infty$ appear in (5.20)–(5.22).)

In the following we discuss the energy of the solutions. First consider the $\tilde{H}$. If the Hamiltonian is $\tilde{H}$, the energy is given by (4.7). Substituting $\lambda_j=\eta_\alpha$ for the real rapidities, $\xi_\alpha \pm \mathrm{i}\gamma$ for the 2-strings, and (5.16) together with (5.17) for the trios and the very wide roots, finally evaluating the sums over the η_α and ξ_α using the root-densities obtained from (5.18) and (5.19) we arrive at

$$\tilde{E} = \tilde{N}\left(N_{1/2}\tilde{\varepsilon}_{1/2} + N_1\tilde{\varepsilon}_1 + \sum_h \tilde{\varepsilon}_{1/2}(\eta_h)\right) \tag{5.24}$$

where

$$\tilde{\varepsilon}_{1/2} = \cot\gamma - 2\int_0^\infty \frac{\sinh(k(\pi-\gamma))}{\cosh(\kappa\gamma)\sinh(\kappa\pi)}\,\mathrm{d}k \tag{5.25}$$

$$\tilde{\varepsilon}_1 = \cot\frac{3\gamma}{2} - 2\int_0^\infty \frac{\sinh(k(\pi-2\gamma))}{\cosh(k\gamma)\sinh(k\pi)}\,\mathrm{d}k \tag{5.26}$$

and

$$\tilde{\varepsilon}_{1/2}(\eta_h) = \frac{\pi}{\gamma\cosh(\pi\eta_h/\gamma)}. \tag{5.27}$$

It is remarkable that neighter the ξ_h nor the χ_n contribute to the energy.

The momentum can be calculated according to (4.13). A straightforward calculation using the distributions of the real roots and 2-strings and the form of the other complex roots gives (for $S^z = 0$)

$$P = \frac{N\pi}{2} + \sum_h 2\tan^{-1}\left(\exp\frac{\pi\eta_h}{\gamma}\right) + \sum_{h'} 2\tan^{-1}\left(\exp\frac{\pi\xi_{h'}}{\gamma}\right). \tag{5.28}$$

Here we have used that $N_{1/2} = N_1 = N$ even. If we denote

$$p_h^{(1/2)} = 2\tan^{-1}\left(\exp\frac{\pi\eta h}{\gamma}\right)$$

$$p_h^{(1)} = 2\tan^{-1}\left(\exp\frac{\pi\xi_h}{\gamma}\right) \tag{5.29}$$

we can write

$$\tilde{E} - \tilde{E}_0 = \tilde{N}\sum_h \frac{\pi}{\gamma}\sin p_h^{(1/2)}. \tag{5.30}$$

We should note, that these states are macroscopically degenerated (in energy but not in momentum), as the $p_h^{(1)}$ do not contribute to the energy.

Calculating the energy according to $\bar{H}$ (4.15) yields

$$\bar{E} = \bar{N}\left(N_{1/2}\bar{\varepsilon}_{1/2} + N_1\bar{\varepsilon}_1 + \sum_h \bar{\varepsilon}_1(\xi_h)\right) \tag{5.31}$$

with

$$\bar{\varepsilon}_{1/2} = \cot\gamma + \cot 2\gamma - 2\int_0^\infty \frac{\sinh(k(\pi-2\gamma))}{\cosh(k\gamma)\sinh(k\pi)}\,\mathrm{d}k \tag{5.32}$$

$$\bar{\varepsilon}_1 = -\cot\frac{\gamma}{2} - 4\int_0^\infty \frac{\sinh(k(\pi-2\gamma))}{\sinh(k\pi)}\,\mathrm{d}k \tag{5.33}$$

and

$$\bar{\varepsilon}_1(\xi_h) = \frac{\pi}{\gamma\cosh(\pi\xi_h/\gamma)}. \tag{5.34}$$

Here the holes in the real η distribution and the non-2-string complex roots possess zero energy. Using (5.29) the excitation energies have the form

$$\bar{E} - \bar{E}_0 = \bar{N}\sum_h \frac{\pi}{\gamma}\sin p_h^{(1)}. \tag{5.35}$$

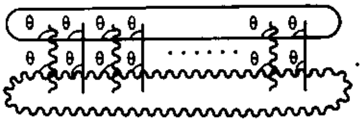
In the above calculations we have seen, that both if $\tilde{H}$ or $\bar{H}$ is the Hamiltonian, the holes in real rapidity distribution (η-holes), and the holes in the 2-string distribution

(ξ-holes) are the elementary excitations. These excitations behave, however, differently: while both carry momentum in both cases, only one of them possesses non-zero energy. Considering a system which is described by a linear combination of $\tilde{H}$ and $\bar{H}$, we can extrapolate between the two limiting cases. The coefficients also can be chosen so that the dispersion is

$$E-E_0=\frac{\pi}{\gamma}\left(\sum_h \sin p_h^{(1/2)}+\sum_{h'} \sin p_{h'}^{(1/2)}\right). \tag{5.36}$$

This system is conformally invariant, and the corresponding Hamiltonian can be derived from the τ-matrix

$$\tau^{(\mathrm{alt})}(\theta)=\tilde{\tau}^{(\mathrm{alt})}(\theta)\bar{\tau}^{(\mathrm{alt})}(\theta): \tag{5.37}$$



Finally we want to prove that the string-hypothesis (5.1) used to solve the Bethe ansatz equations holds. This we do so that we show the corrections to (5.1) are exponentially small. Setting

$$\lambda_j=\xi_\alpha+\mathrm{i}(\gamma/2+\delta_\alpha) \tag{5.38}$$

we find

$$\mathrm{Im}\left[\mathrm{i}\ln\frac{\sin(2\gamma+\delta)}{\sin\delta}+N_{1/2}\varphi_{1/2}(\xi)+N_1\varphi_1(\xi)\right]=O(1). \tag{5.39}$$

Here we assume $\delta \ll 1$, $O(1)$ means terms of the order unity, and we use the notation

$$\varphi_{1/2}(\xi)=-\phi(\xi+\mathrm{i}\gamma/2,\gamma/2)+\int \mathrm{d}\eta\,\rho_{1/2}(\eta)\phi(\xi+\mathrm{i}\gamma/2-\eta,\gamma) \tag{5.40}$$

and

$$\varphi_1(\xi)=-\phi(\xi+\mathrm{i}\gamma/2,\gamma)+\int \mathrm{d}\xi'\,\rho_1(\xi')\times[\phi(\xi+\mathrm{i}\gamma/2-\xi',\gamma/2)+\phi(\xi+\mathrm{i}\gamma/2-\xi',3\gamma/2)]. \tag{5.41}$$

It is clear that for low excited states $\rho_{1/2}$ and ρ_1 can be replaced by the ground-state distributions $\rho_{1/2}^0$ and ρ_1^0. Evaluating the integrals we find

$$\mathrm{Im}\,\varphi_1(\xi)=0 \qquad \mathrm{Im}\,\varphi_{1/2}(\xi)=\ln\left(\tanh\frac{\pi\xi}{2\gamma}\right) \tag{5.42}$$

so

$$\frac{\sin|\delta|}{\sin(2\gamma)}=\left|\tanh\frac{\pi\xi}{2\gamma}\right|^{N_{1/2}} \tag{5.43}$$

i.e. δ is indeed exponentially small in N.

References

[1] Bethe H 1931 *Z. Phys.* **71** 205
Orbach R 1958 *Phys. Rev.* **112** 309
Walker L R 1959 *Phys. Rev.* **116** 1089

[2] Fateev V A and Zamolodchikov A B 1980 *Sov. J. Nucl. Phys.* **32** 298
[3] Kulish P P and Reshetikhin N Yu 1981 *Soviet Math.* **101** 2435
[4] de Vega H J 1989 *Int. J. Mod. Phys.* A **4** 2371; 1990 *Nucl. Phys.* B *(Proc. Suppl.)* **18A** 229
[5] de Vega H J and Lopes E 1991c *Phys. Rev. Lett.* **67** 489; 1991b *Nucl. Phys.* B **362** 261; 1991c *Preprint* LPTHE 91/29
[6] Babujian H M and Tsevelik A M 1986 *Nucl. Phys.* B **265** 24
[7] Takhtadzhyan L A 1982 *Phys. Lett.* **87A** 479
Babujian H M 1983 *Nucl Phys.* B **215** 317
[8] Woynarovich F 1982 *J. Phys. A: Math. Gen.* **15** 2985
Destri C and Lowenstein J H 1982 *Nucl. Phys.* B **205** 369
Babelon O, de Vega H J and Viallet C M 1983 *Nucl. Phys.* B **220** 13
[9] Destri C and de Vega H J 1987 *Nucl. Phys.* B **290** 363; 1989 *J. Phys. A: Math. Gen.* **22** 1329
[10] de Vega H J and Woynarovich F 1990 *J. Phys. A: Math. Gen.* **23** 1613